1. *Let $f : [0,1] \to \mathbb{R}$ be a bounded function, and suppose the lower integral of f on $[0,1]$ is positive. Prove that there exists an interval $[r_1, r_2] \subseteq [0,1]$, $r_1 \neq r_2$, such that $f(x) > 0$ for all x in $[r_1, r_2]$.*

   Solution: If the lower integral of f is positive then the Riemann integral of f is positive. Suppose $f = 0$ on $[0,1]$ implies that $\int_0^1 f = 0$. Therefore $\int_0^1 f$ is positive implies that f is positive on some interval $[r_1, r_2] \subseteq [0,1]$.

   $\square$

2. *Let f be a Riemann integrable function on $[a,b]$, $a < b$, and let $F(x) = \int_a^x f(t)dt$, $x \in [a,b]$. Prove that F is continuous on [a,b].*

   Solution: We can find the proof in the book 'Elementary Analysis' by Kenneth A. Ross. Theorem 34.3, Page - 294. $\square$

3. *Find the limit (if exists):* $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$

   Solution: Let $h(x,y) = \frac{x^2y}{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2 \setminus 0$. Let $\epsilon > 0$ be given, choose $\delta > 0$ such that $\delta = \epsilon$. Whenever $0 < (x,y) \in \mathbb{R}^2$ with $|(x,y)| := \max\{|x|, |y|\} < \epsilon$, we have

$$|h(x,y) - (0,0)| = |h(x,y)| = |\frac{x^2y}{x^2+y^2}| \leq \frac{x^2|y|}{x^2} = |y| < \delta < \epsilon.$$

   Thus $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = (0,0)$. $\square$

4. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a C^∞-function, and let*

$$g(x_1, ..., x_n) = f(e^{x_1}, ..., e^{x_n}),$$

   for all $(x_1, ..., x_n) \in \mathbb{R}^n$. Suppose that

$$\sum_{i=1}^{n}(x_i^2 \frac{\partial^2 f}{\partial x_i^2} + x_i \frac{\partial f}{\partial x_i}) = 0.$$

   Compute $\sum\limits_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}$.

   Solution: Given that

$$g(x_1, ..., x_n) = f(e^{x_1}, ..., e^{x_n}).$$

   Compute $\frac{\partial g}{\partial x_i}$,

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial e^{x_i}} e^{x_i}.$$

Compute $\frac{\partial^2 g}{\partial x_i^2}$,

$$\frac{\partial^2 g}{\partial x_i^2} = \frac{\partial f}{\partial e^{x_i}} e^{x_i} + {e^{x_i}}^2 \frac{\partial^2 f}{\partial {e^{x_i}}^2}.$$

Using the assumption $\sum\limits_{i=1}^{n} (x_i^2 \frac{\partial^2 f}{\partial x_i^2} + x_i \frac{\partial f}{\partial x_i}) = 0$ and by above computation $\sum\limits_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2} = 0$.

□

5. *Which of the following statements are true, and which are false? Justify your answer.*
   (i). $\overline{B_r(a)} = \{x \in X : d(x,a) \leq r\}$.
   (ii). If every subset of X is compact, then X is a finite set.
   (iii). Interior of a connected subset of X is connected.

   Solution: (i). It is not necessarily true that the closure of the open ball $B_r(a)$ is equal to the closed ball of the same radius r centred at the same point a. For example, take X to be any set and define a metric

$$d(x,y) = \begin{cases} 0 & \text{if and only if } \ x = y \\ 1 & \text{otherwise} \end{cases}$$

   The open unit ball of radius 1 around any point a is a singleton set $\{x\}$. Its closure is also the singleton set. However the closed unit ball of radius 1 is everything.

   (ii). Since metric space is Hausdroff, (ii) is true. If X is compact, every subset has a limit point. Suppose X is not finite then there exists an infinite subset A. We can choose a limit point x of A, take it away from A if it is in A and denote the new set by A'. It is still infinite. By assumption A' is compact, since X is Hausdroff, A' must be closed, but there exists a limit point of A' that is not in A'. This is a contradiction.

   (iii). This is need not be true. If $X \subset \mathbb{R}^2$ is the union of two closed disks of radius 1, one with centre at $(1,0)$ and another with centre at $(-1,0)$ then X is connected but its interior is not. □

6. *Let X be a compact metric space, and let $f : X \to X$ be a function. suppose that*

$$d(f(x), f(y)) < d(x,y)$$

   for all $x \neq y$. Prove that f has a unique fixed point.

   Solution: We can find the proof in the book 'Topology of Metric Spaces' by S. Kumaresan. Theorem 6.4.5, Page - 143. □

7. *Let X be a complete countable metric space. Prove that there exists an element $x \in X$ such that $\{x\}$ is open.*

   Solution: To prove that there exists an element $x \in X$ such that $\{x\}$ is open, it is enough to prove that X has a isolated point. Suppose X is a complete metric space with no isolated points. Since X is countable and fix an enumeration $X = \{x_n : n \in \mathbb{N}\}$. For each $x \in X$, let $O_x := X \setminus \{x\}$. By our assumption O_{x_n} is dense in X. Using Baire category theorem, we have $\cap_{n\in\mathbb{N}} O_{x_n}$ is dense in (X,d). But this is a contradiction, because $\cap_{n\in\mathbb{N}} O_{x_n}$ is empty. □

8. *Determine the nature of the critical points of* $f(x,y) = 2x^3 - 6xy + y^2 + 4y,\ (x,y) \in \mathbb{R}^2$.

Solution: We will first to get all the first and second order derivatives

$$f_x = 6x^2 - 6y, f_y = -6x + 2y + 4, f_{xx} = 6x, f_{yy} = 2, f_{xy} = -6.$$

We can solve the first equation for y as

$$6x^2 - 6y = 0 \Rightarrow y = x^2.$$

Plugging this into the second equation gives,

$$x^2 - 3x + 4 = 0.$$

From this we can see that we must have $x = 1$ or $x = 2$. Now use the fact that $y = x^2$

$$x = 1, y = 1 \Rightarrow (1,1)$$

$$x = 2, y = 4 \Rightarrow (2,4)$$

So we get two critical points. All we need to do now is classify them. To do this we will need D. Here is the general formula for D.

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 6x2 - (-6)^2 = 12x - 36.$$

To classify the critical points all that we need to do is plug the critical points and use the fact above to classify them.

$$D(1,1) = 12 - 36 = -24 < 0$$

So, for $(1,1)$, D is negative and so this must be a saddle point.

$$D(2,4) = 24 - 36 = -12 < 0$$

For $(2,4)$, D is negative so this is also a saddle point.

□

9. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a function. Suppose that* f *is a differentiable at* $(0,0)$ *and*

$$\lim_{x\to 0} \frac{f(x,x) - f(x,-x)}{x} = 1.$$

Compute $\frac{\partial f}{\partial y}(0,0)$.

Solution:

$$\lim_{x\to 0} \frac{f(x,x) - f(x,-x)}{x} = 1$$

$$\lim_{x\to 0} \frac{f(x,x) - f(0,0)}{x} - \lim_{x\to 0} \frac{f(x,-x) - f(0,0)}{x} = 1$$

$$\lim_{x\to 0} \frac{f(x(1,1)) - f(0,0)}{x} - \lim_{x\to 0} \frac{f(x(1,-1)) - f(0,0)}{x} = 1$$

Using the definition of derational derivative, we have

$$\nabla f|_{(0,0)}(1,1) - \nabla f|_{(0,0)}(1,-1) = 1$$

$$
\begin{aligned}
\nabla f|_{(0,0)}(0,2) &= 1 \\
2\nabla f|_{(0,0)}(0,1) &= 1 \\
\nabla f|_{(0,0)}(0,1) &= \frac{1}{2} \\
\frac{\partial f}{\partial y}(0,0) &= \frac{1}{2}.
\end{aligned}
$$

□